RATIONAL ONE-STEP METHOD OF ORDER SEVEN FOR INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

A rational one-step numerical integrator of order seven is proposed for initial value problems in ordinary differential equations. it compares very favourably with earlier works.

Introduction

We will consider the initial value problem (IVP)

$$y' = f(x, y),\ y(a) = y_0\ ;\ y, f \in R^m \ \ and\ \ x \in [a, b],\ a, b \in R \qquad (1.1)$$

whose solution may contain singularities. The points of singularities of the solution of the IVP (1.1) are the poles of the rational interpolant.

Existing schemes for the numerical solution of (1.1) include those of Lambert and Shaw (1965), Luke et al (1975), Van Niekerk (1987), Lambert (1974), Shaw (1967) and Fatunla (1982, 1986, 1988, 1990, 1994), Fatunla and Aashikpelokai (1994), Otunta and Ikhile (1996, 1999) and Otunta and Nwachukwu (2003, 2005). These schemes are based on rational function interpolation since rational functions are more accurate than polynomials in representing a function in the neighbourhood of singularities.

Niekerk (1987), Otunta and Ikhile (1999), Otunta and Nwachukwu (2003) and Otunta and Nwachukwu (2005) constructed schemes of order three, four, five and six respectively.

In this paper, we will develop a one-step rational integrator of order seven with a better accuracy of solution.

Derivation of the Integrator

We interpolate the theoretical solution of (1.1) by

$$y(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}$$

(2.1)

The resultant one step scheme is given by

$$y_{n+1} = \frac{a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3}{1 + b_1 x_{n+1} + b_2 x_{n+1}^2 + b_3 x_{n+1}^3 + b_4 x_{n+1}^4}$$

(2.2)

Writing (2.2) as

$$y_{n+1} = \left(a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3\right)\left[1 + \sum_{r=1}^{\infty}(-1)^r\left(\sum_{j=1}^{4} b_j x_{n+1}^j\right)^r\right]$$

(2.3)

and superimposing it on

$$y(x_{n+1}) = \sum_{j=0}^{\infty} \frac{h^j y_n^{(j)}}{j!}\ ,\ y_n^{(0)} = y_n$$

(2.4)

we obtain the method parameters:

$$a_0 = y_n$$

$$a_1 = y_n b_1 + \frac{h\, y_n^{\prime}}{x_{n+1}}$$

$$a_2 = y_n b_2 + \frac{h y_n^{I}}{x_{n+1}} b_1 + \frac{h^2 y_n^{II}}{2! x_{n+1}^2}$$

(2.5)

$$a_3 = y_n b_3 + \frac{h y_n^{I}}{x_{n+1}} b_2 + \frac{h^2 y_n^{II}}{2! x_{n+1}^2} b_1 + \frac{h^3 y_n^{III}}{3! x_{n+1}^3}$$

$$b_1 = \frac{T_1}{T}, \quad b_2 = \frac{T_2}{T}, \quad b_3 = \frac{T_3}{T}, \quad b_4 = \frac{T_4}{T}$$

where

$$T = \frac{h^3 y_n^{III}}{3! x_{n+1}^3}\left\{ \frac{h^3 y_n^{III}}{3! x_{n+1}^3}\left(\frac{h^6 y_n^{III} y_n^{III}}{3!\,3! x_{n+1}^6} - \frac{h^6 y_n^{II} y_n^{(4)}}{2!\,4! x_{n+1}^6}\right) - \frac{h^2 y_n^{II}}{2! x_{n+1}^2}\left(\frac{h^7 y_n^{III} y_n^{(4)}}{3!\,4! x_{n+1}^7} - \frac{h^7 y_n^{II} y_n^{(5)}}{2!\,5! x_{n+1}^7}\right) + \frac{h y_n^{I}}{x_{n+1}}\left(\frac{h^8 y_n^{(4)} y_n^{(4)}}{4!\,4! x_{n+1}^8} - \frac{h^8 y_n^{III} y_n^{(5)}}{3!\,5! x_{n+1}^8}\right)\right\}$$

$$- \frac{h^2 y_n^{II}}{2! x_{n+1}^2}\left\{ \frac{h^4 y_n^{(4)}}{4! x_{n+1}^4}\left(\frac{h^6 y_n^{III} y_n^{III}}{3!\,3! x_{n+1}^6} - \frac{h^6 y_n^{II} y_n^{(4)}}{2!\,4! x_{n+1}^6}\right) - \frac{h^2 y_n^{II}}{2! x_{n+1}^2}\left(\frac{h^8 y_n^{III} y_n^{(5)}}{3!\,5! x_{n+1}^8} - \frac{h^8 y_n^{II} y_n^{(6)}}{2!\,6! x_{n+1}^8}\right) + \frac{h y_n^{I}}{x_{n+1}}\left(\frac{h^9 y_n^{(4)} y_n^{(5)}}{4!\,5! x_{n+1}^9} - \frac{h^9 y_n^{III} y_n^{(6)}}{3!\,6! x_{n+1}^9}\right)\right\}$$

$$+ \frac{h y_n^{I}}{x_{n+1}}\left\{ \frac{h^4 y_n^{(4)}}{4! x_{n+1}^4}\left(\frac{h^7 y_n^{III} y_n^{(4)}}{3!\,4! x_{n+1}^7} - \frac{h^7 y_n^{II} y_n^{(5)}}{2!\,5! x_{n+1}^7}\right) - \frac{h^3 y_n^{III}}{3! x_{n+1}^3}\left(\frac{h^8 y_n^{(5)} y_n^{III}}{3!\,5! x_{n+1}^8} - \frac{h^8 y_n^{II} y_n^{(6)}}{2!\,6! x_{n+1}^8}\right) + \frac{h y_n^{I}}{x_{n+1}}\left(\frac{h^{10} y_n^{(5)} y_n^{(5)}}{5!\,5! x_{n+1}^{10}} - \frac{h^{10} y_n^{(4)} y_n^{(6)}}{4!\,6! x_{n+1}^{10}}\right)\right\}$$

(2.6)

$$- y_n\left\{ \frac{h^4 y_n^{(4)}}{4! x_{n+1}^4}\left(\frac{h^8 y_n^{(4)} y_n^{(4)}}{4!\,4! x_{n+1}^8} - \frac{h^8 y_n^{III} y_n^{(5)}}{3!\,5! x_{n+1}^8}\right) - \frac{h^3 y_n^{III}}{3! x_{n+1}^3}\left(\frac{h^9 y_n^{(4)} y_n^{(5)}}{4!\,5! x_{n+1}^9} - \frac{h^9 y_n^{III} y_n^{(6)}}{3!\,6! x_{n+1}^9}\right) + \frac{h^2 y_n^{II}}{2! x_{n+1}^2}\left(\frac{h^{10} y_n^{(5)} y_n^{(5)}}{5!\,5! x_{n+1}^{10}} - \frac{h^{10} y_n^{(4)} y_n^{(6)}}{4!\,6! x_{n+1}^{10}}\right)\right\}$$

$$T_1 = \frac{h^4 y_n^{(4)}}{4! x_{n+1}^4}\left\{ \frac{h^3 y_n^{III}}{3! x_{n+1}^3}\left(\frac{h^6 y_n^{III} y_n^{III}}{3!\,3! x_{n+1}^6} - \frac{h^6 y_n^{II} y_n^{(4)}}{2!\,4! x_{n+1}^6}\right) - \frac{h^2 y_n^{II}}{2! x_{n+1}^2}\left(\frac{h^7 y_n^{III} y_n^{(4)}}{3!\,4! x_{n+1}^7} - \frac{h^7 y_n^{II} y_n^{(5)}}{2!\,5! x_{n+1}^7}\right) + \frac{h y_n^{I}}{x_{n+1}}\left(\frac{h^8 y_n^{(4)} y_n^{(4)}}{4!\,4! x_{n+1}^8} - \frac{h^8 y_n^{III} y_n^{(5)}}{3!\,5! x_{n+1}^8}\right)\right\}$$

$$- \frac{h^2 y_n^{II}}{2! x_{n+1}^2}\left\{ -\frac{h^5 y_n^{(5)}}{5! x_{n+1}^5}\left(\frac{h^6 y_n^{III} y_n^{III}}{3!\,3! x_{n+1}^6} - \frac{h^6 y_n^{II} y_n^{(4)}}{2!\,4! x_{n+1}^6}\right) - \frac{h^2 y_n^{II}}{2! x_{n+1}^2}\left(-\frac{h^9 y_n^{III} y_n^{(6)}}{3!\,6! x_{n+1}^9} + \frac{h^9 y_n^{II} y_n^{(VII)}}{2!\,7! x_{n+1}^9}\right) + \frac{h y_n^{I}}{x_{n+1}}\left(-\frac{h^{10} y_n^{(4)} y_n^{(6)}}{4!\,6! x_{n+1}^{10}} + \frac{h^{10} y_n^{III} y_n^{(7)}}{3!\,7! x_{n+1}^{10}}\right)\right\}$$

$$
+\frac{hy_n'}{x_{n+1}}\left\{\begin{array}{l}
-\frac{h^5y_n^{(5)}}{5!\,x_{n+1}^5}\left(\frac{h^7y_n'''y_n^{(4)}}{3!\,4!\,x_{n+1}^7}-\frac{h^7y_n''y_n^{(5)}}{2!\,5!\,x_{n+1}^7}\right)-\frac{h^3y_n'''}{3!\,x_{n+1}^3}\left(-\frac{h^9y_n'''y_n^{(6)}}{3!\,6!\,x_{n+1}^9}+\frac{h^9y_n''y_n^{(7)}}{2!\,7!\,x_{n+1}^9}\right)\\
+\frac{hy_n'}{x_{n+1}}\left(-\frac{h^{11}y_n^{(5)}y_n^{(6)}}{5!\,6!\,x_{n+1}^{11}}+\frac{h^{11}y_n^{(4)}y_n^{(7)}}{4!\,7!\,x_{n+1}^{11}}\right)
\end{array}\right\}
$$

(2.7)

$$
-y_n\left\{\begin{array}{l}
-\frac{h^5y_n^{(5)}}{5!\,x_{n+1}^5}\left(\frac{h^8y_n^{(4)}y_n^{(4)}}{4!\,4!\,x_{n+1}^8}-\frac{h^8y_n'''y_n^{(5)}}{3!\,5!\,x_{n+1}^8}\right)-\frac{h^3y_n'''}{3!\,x_{n+1}^3}\left(-\frac{h^{10}y_n^{(4)}y_n^{(6)}}{4!\,6!\,x_{n+1}^{10}}+\frac{h^{10}y_n'''y_n^{(7)}}{3!\,7!\,x_{n+1}^{10}}\right)\\
+\frac{h^2y_n''}{2!\,x_{n+1}^2}\left(-\frac{h^{11}y_n^{(5)}y_n^{(6)}}{5!\,6!\,x_{n+1}^{11}}+\frac{h^{11}y_n^{(4)}y_n^{(7)}}{4!\,7!\,x_{n+1}^{11}}\right)
\end{array}\right\}
$$

$$
T_2=\frac{h^3y_n'''}{3!\,x_{n+1}^3}\left\{\begin{array}{l}
-\frac{h^5y_n^{(5)}}{5!\,x_{n+1}^5}\left(\frac{h^6y_n'''y_n'''}{3!\,3!\,x_{n+1}^6}-\frac{h^6y_n''y_n^{(4)}}{2!\,4!\,x_{n+1}^6}\right)-\frac{h^2y_n''}{2!\,x_{n+1}^2}\left(-\frac{h^9y_n'''y_n^{(6)}}{3!\,6!\,x_{n+1}^9}+\frac{h^9y_n''y_n^{(7)}}{2!\,7!\,x_{n+1}^9}\right)\\
+\frac{hy_n'}{x_{n+1}}\left(-\frac{h^{10}y_n^{(4)}y_n^{(6)}}{4!\,6!\,x_{n+1}^{10}}+\frac{h^{10}y_n'''y_n^{(7)}}{3!\,7!\,x_{n+1}^{10}}\right)
\end{array}\right\}
$$

$$
+\frac{h^4y_n^{(4)}}{4!\,x_{n+1}^4}\left\{\begin{array}{l}
\frac{h^4y_n^{(4)}}{4!\,x_{n+1}^4}\left(\frac{h^6y_n'''y_n'''}{3!\,3!\,x_{n+1}^6}-\frac{h^6y_n''y_n^{(4)}}{2!\,4!\,x_{n+1}^6}\right)-\frac{h^2y_n''}{2!\,x_{n+1}^2}\left(\frac{h^8y_n'''y_n^{(5)}}{3!\,5!\,x_{n+1}^8}-\frac{h^8y_n''y_n^{(6)}}{2!\,6!\,x_{n+1}^8}\right)\\
+\frac{hy_n'}{x_{n+1}}\left(\frac{h^9y_n^{(4)}y_n^{(5)}}{4!\,5!\,x_{n+1}^9}-\frac{h^9y_n'''y_n^{(6)}}{3!\,6!\,x_{n+1}^9}\right)
\end{array}\right\}
$$

(2.8)

$$
+\frac{hy_n'}{x_{n+1}}\left\{\begin{array}{l}
\frac{h^4y_n^{(4)}}{4!\,x_{n+1}^4}\left(-\frac{h^9y_n'''y_n^{(6)}}{3!\,6!\,x_{n+1}^9}+\frac{h^9y_n''y_n^{(7)}}{2!\,7!\,x_{n+1}^9}\right)+\frac{h^5y_n^{(5)}}{5!\,x_{n+1}^5}\left(\frac{h^8y_n'''y_n^{(5)}}{3!\,5!\,x_{n+1}^8}-\frac{h^8y_n''y_n^{(6)}}{2!\,6!\,x_{n+1}^8}\right)\\
+\frac{hy_n'}{x_{n+1}}\left(-\frac{h^{12}y_n^{(5)}y_n^{(7)}}{5!\,7!\,x_{n+1}^{12}}+\frac{h^{12}y_n^{(6)}y_n^{(6)}}{6!\,6!\,x_{n+1}^{12}}\right)
\end{array}\right\}
$$

$$
-y_n\left\{\begin{array}{l}
\frac{h^4y_n^{(4)}}{4!\,x_{n+1}^4}\left(-\frac{h^{10}y_n^{(4)}y_n^{(6)}}{4!\,6!\,x_{n+1}^{10}}+\frac{h^{10}y_n'''y_n^{(7)}}{3!\,7!\,x_{n+1}^{10}}\right)+\frac{h^5y_n^{(5)}}{5!\,x_{n+1}^5}\left(\frac{h^9y_n^{(4)}y_n^{(5)}}{4!\,5!\,x_{n+1}^9}-\frac{h^9y_n'''y_n^{(6)}}{3!\,6!\,x_{n+1}^9}\right)\\
+\frac{h^2y_n''}{2!\,x_{n+1}^2}\left(-\frac{h^{12}y_n^{(5)}y_n^{(7)}}{5!\,7!\,x_{n+1}^{12}}+\frac{h^{12}y_n^{(6)}y_n^{(6)}}{6!\,6!\,x_{n+1}^{12}}\right)
\end{array}\right\}
$$

$$
T_3=\frac{h^3y_n'''}{3!\,x_{n+1}^4}\left\{\begin{array}{l}
\frac{h^3y_n'''}{3!\,x_{n+1}^3}\left(-\frac{h^9y_n'''y_n^{(6)}}{3!\,6!\,x_{n+1}^9}+\frac{h^9y_n''y_n^{(7)}}{2!\,7!\,x_{n+1}^9}\right)+\frac{h^5y_n^{(5)}}{5!\,x_{n+1}^5}\left(\frac{h^7y_n'''y_n^{(4)}}{3!\,4!\,x_{n+1}^7}-\frac{h^7y_n''y_n^{(5)}}{2!\,5!\,x_{n+1}^7}\right)\\
+\frac{hy_n'}{x_{n+1}}\left(-\frac{h^{11}y_n^{(4)}y_n^{(7)}}{4!\,7!\,x_{n+1}^{11}}+\frac{h^{11}y_n^{(5)}y_n^{(6)}}{5!\,6!\,x_{n+1}^{11}}\right)
\end{array}\right\}
$$

$$-\frac{h^2 y_n^{II}}{2!x_{n+1}^2}\left\{\begin{array}{l}\frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}\left(-\frac{h^9 y_n^{III} y_n^{(6)}}{3!\,6!x_{n+1}^9}+\frac{h^9 y_n^{II} y_n^{(7)}}{2!\,7!\,x_{n+1}^9}\right)+\frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}\left(\frac{h^8 y_n^{III} y_n^{(5)}}{3!\,5!x_{n+1}^8}-\frac{h^8 y_n^{II} y_n^{(6)}}{2!\,6!x_{n+1}^8}\right)\\ +\frac{h y_n^{I}}{x_{n+1}}\left(-\frac{h^{12} y_n^{(5)} y_n^{(7)}}{5!\,7!x_{n+1}^{12}}+\frac{h^{12} y_n^{(6)} y_n^{(6)}}{6!\,6!x_{n+1}^{12}}\right)\end{array}\right\}$$

$$-\frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}\left\{\begin{array}{l}\frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}\left(\frac{h^7 y_n^{III} y_n^{(4)}}{3!\,4!x_{n+1}^7}-\frac{h^7 y_n^{II} y_n^{(5)}}{2!\,5!\,x_{n+1}^7}\right)-\frac{h^3 y_n^{III}}{3!x_{n+1}^3}\left(\frac{h^8 y_n^{III} y_n^{(5)}}{3!\,5!\,x_{n+1}^8}-\frac{h^8 y_n^{II} y_n^{(6)}}{2!\,6!\,x_{n+1}^8}\right)\\ +\frac{h y_n^{I}}{x_{n+1}}\left(\frac{h^{10} y_n^{(5)} y_n^{(5)}}{5!\,5!x_{n+1}^{10}}-\frac{h^{10} y_n^{(4)} y_n^{(6)}}{4!\,6!x_{n+1}^{10}}\right)\end{array}\right\}$$

(2.9)

$$-y_n\left\{\begin{array}{l}\frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}\left(-\frac{h^{11} y_n^{(4)} y_n^{(7)}}{4!\,7!x_{n+1}^{11}}+\frac{h^{11} y_n^{(5)} y_n^{(6)}}{5!\,6!\,x_{n+1}^{11}}\right)-\frac{h^3 y_n^{III}}{3!x_{n+1}^3}\left(-\frac{h^{12} y_n^{(5)} y_n^{(7)}}{5!\,7!x_{n+1}^{12}}+\frac{h^{12} y_n^{(6)} y_n^{(6)}}{6!\,6!x_{n+1}^{12}}\right)\\ -\frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}\left(\frac{h^{10} y_n^{(5)} y_n^{(5)}}{5!\,5!x_{n+1}^{10}}-\frac{h^{10} y_n^{(4)} y_n^{(6)}}{4!\,6!x_{n+1}^{10}}\right)\end{array}\right\}$$

$$T_1=\frac{h^3 y_n^{III}}{3!\,x_{n+1}^3}\left\{\begin{array}{l}\frac{h^3 y_n^{III}}{3!x_{n+1}^3}\left(-\frac{h^{10} y_n^{III} y_n^{(7)}}{3!\,7!x_{n+1}^{10}}+\frac{h^{10} y_n^{(4)} y_n^{(6)}}{4!\,6!\,x_{n+1}^{10}}\right)-\frac{h^2 y_n^{II}}{2!x_{n+1}^2}\left(-\frac{h^{11} y_n^{(4)} y_n^{(7)}}{4!\,7!x_{n+1}^{11}}+\frac{h^{11} y_n^{(5)} y_n^{(6)}}{5!\,6!x_{n+1}^{11}}\right)\\ -\frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}\left(\frac{h^8 y_n^{(4)} y_n^{(4)}}{4!\,4!x_{n+1}^8}-\frac{h^8 y_n^{III} y_n^{(5)}}{3!\,5!x_{n+1}^8}\right)\end{array}\right\}$$

$$-\frac{h^2 y_n^{II}}{2!\,x_{n+1}^2}\left\{\begin{array}{l}\frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}\left(-\frac{h^{10} y_n^{III} y_n^{(7)}}{3!\,7!x_{n+1}^{10}}+\frac{h^{10} y_n^{(4)} y_n^{(6)}}{4!\,6!\,x_{n+1}^{10}}\right)-\frac{h^2 y_n^{II}}{2!x_{n+1}^2}\left(-\frac{h^{12} y_n^{(5)} y_n^{(7)}}{5!\,7!x_{n+1}^{12}}+\frac{h^{12} y_n^{(6)} y_n^{(6)}}{6!\,6!x_{n+1}^{12}}\right)\\ -\frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}\left(\frac{h^9 y_n^{(4)} y_n^{(5)}}{4!\,5!x_{n+1}^9}-\frac{h^9 y_n^{III} y_n^{(6)}}{3!\,6!x_{n+1}^9}\right)\end{array}\right\}$$

(2.10)

$$+\frac{h y_n^{I}}{x_{n+1}}\left\{\begin{array}{l}\frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}\left(-\frac{h^{11} y_n^{(4)} y_n^{(7)}}{4!\,7!x_{n+1}^{11}}+\frac{h^{11} y_n^{(5)} y_n^{(6)}}{5!\,6!\,x_{n+1}^{11}}\right)-\frac{h^3 y_n^{III}}{3!x_{n+1}^3}\left(-\frac{h^{12} y_n^{(5)} y_n^{(7)}}{5!\,7!\,x_{n+1}^{12}}+\frac{h^{12} y_n^{(6)} y_n^{(6)}}{6!\,6!\,x_{n+1}^{12}}\right)\\ -\frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}\left(\frac{h^{10} y_n^{(5)} y_n^{(5)}}{5!\,5!x_{n+1}^{10}}-\frac{h^{10} y_n^{(4)} y_n^{(6)}}{4!\,6!x_{n+1}^{10}}\right)\end{array}\right\}$$

$$+\frac{h^4 y_n^{(4)}}{4!\,x_{n+1}^4}\left\{\begin{array}{l}\frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}\left(\frac{h^8 y_n^{(4)} y_n^{(4)}}{4!\,4!x_{n+1}^8}-\frac{h^8 y_n^{III} y_n^{(5)}}{3!\,5!\,x_{n+1}^8}\right)-\frac{h^3 y_n^{III}}{3!x_{n+1}^3}\left(\frac{h^9 y_n^{(4)} y_n^{(5)}}{4!\,5!x_{n+1}^9}-\frac{h^9 y_n^{III} y_n^{(6)}}{3!\,6!x_{n+1}^9}\right)\\ +\frac{h^2 y_n^{II}}{2!x_{n+1}^2}\left(\frac{h^{10} y_n^{(5)} y_n^{(5)}}{5!\,5!x_{n+1}^{10}}-\frac{h^{10} y_n^{(4)} y_n^{(6)}}{4!\,6!x_{n+1}^{10}}\right)\end{array}\right\}$$

The error on truncation is

$$Lte_n = x_{n+1}^{8}\left\{a_0 \sum_{j=1}^{7} C_8^{(j)}(-)^{j} + a_1 \sum_{j=1}^{6} C_7^{(j)}(-)^{j} + a_2 \sum C_6(-)^{j} + a_3 \sum C_5(\ldots) \text{[illegible]}\right.$$

$$-\frac{h^8 y_n^{(8)}}{8!} + 0(h^9)$$

(2.11)

The attainable order of the scheme is seven.

Stability Function

On the application of the scheme on the scalar test problem

$y' = \lambda y,\ Re(\lambda) < 0$ 3

we obtain the stability function

$$\mu(z) = \frac{840 + 360z + 60z^2 + 4z^3}{840 - 480z + 120z^2 - 16z^3 + z^4}$$

(3.2)

we notice that the above stability polynomial satisfies the conditions of L-stability i.e

$|\mu(z)| \leq 1, \quad Re(z) < 0$

$\lim_{z \to \infty} \mu(z) = 0$

(3.3)

Numerical Experiment

Numerical results on three problems are presented.

Problem (1)

Fatunla (1986), Niekerk (1987), Lambert and Shaw (1965), Luke et al (1975), Otunta and Ikhile (1999), Otunta and Nwachukwu (2003), Otunta and Nwachukwu (2005)

$y^1 = 1 + y^2,\ y(0) = 1, \quad 0 \leq x \leq 1$

$$y(x) = tan\left(x + \frac{\pi}{4}\right) \quad (4.1)$$

Table 1.1: Performance of Some Existing Algorithms on the IVP (4.1): $h = 0.05$ [illegible]

/Error/

x	Theoretical Solution	Fatunla (1986)					
		$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = $ [illegible]
0.00	1.000000000						
0.10	1.223048880	1.228 (-2)	8.277 (-4)				
0.20	1.508497647	2.99 (-2)	1.894 (-3)	2.298 (-4)	5.93 (-5)		
0.30	1.895765123	5.580 (-2)	3.520 (-3)	5.507 (-4)	5.434 (-5)	2.363 (-5)	
0.40	2.464962757	1.045 (-1)	6.473 (-3)	6.726 (-4)	1.681 (-4)	2.408 (-5)	
0.50	3.408223442	2.134 (-1)	1.312 (-2)	1.754 (-3)	2.238 (-4)	7.234 (-5)	[illegible]
0.60	5.33185223	5.562 (-1)	3.405 (-2)	4.206 (-3)	7.541 (-4)	1.272 (-4)	[illegible]
0.65	1.340436575	1.160 (0)	6.664 (-2)	8.379 (-3)	1.509 (-3)	2.338 (-4)	[illegible]
0.70	11.681373800	3.092 (0)	1.762 (-1)	2.016 (-2)	3.322 (-3)	7.849 (-4)	7.349 (-5)
0.75	28.238252850	2.903 (1)	1.074 (-1)	1.181 (-1)	1.893 (-2)	4.902 (-3)	1.017 (-4)
0.80	-68.479668346	3.776 (1)	8.284 (0)	[illegible]	1.282 (-1)	2.622 (-2)	1.585 (-2)
0.90	-6.87629546	1.166 (0)	1.952 (-1)	6.093 (-3)	2.108 (-2)	6.158 (-3)	4.264 (-2)
1.00	-4.588037825	3.421 (-1)	2.34 (-1)	9.291 (-3)	1.610 (-2)	1.054 (-2)	[illegible]

Table 1.2: Performance of Some Existing Algorithms on the IVP (4.1); $h = 0.05$, $0 \le x \le 1$, /Error/

x	Theoretical Solution	Van Nickerk (1987)		Lambert and Shaw (1965)	Luke et al (1975)	Otunta & Ikhile (1999)
		$p=1$	$p=2$			
0.00	1.000000000					
0.10	1.223048880	1.0 (-4)	2.0 (-6)	8 (-8)	2 (-4)	2.420 (-7)
0.20	1.508497647	4.0 (-4)	8.0 (-7)	8 (-7)	5 (-4)	2.893 (-7)
0.30	1.895765123	1.0 (-3)	2.0 (-5)	4 (-7)	1 (-3)	6.972 (-7)
0.40	2.464962757	2.0 (-3)	5.0 (-5)	7 (-7)	2 (-3)	1.601 (-6)
0.50	3.408223442	5.0 (-3)	1.0 (-4)	2 (-6)	5 (-3)	3.970 (-6)
0.60	5.33185223	1.0 (2)	5.0 (-4)	4 (-6)	1 (-2)	1.562 (-5)
0.65	1.340436575	3.0 (-3)	1.0 (-3)	8 (-6)	2 (-2)	2.803 (-5)
0.70	11.681373800	7.0 (-2)	3.0 (-3)	2 (-5)	8 (-2)	6.866 (-5)
0.75	28.238252850	5.0 (-1)	2.0 (-2)	2 (-2)	4 (-1)	4.867 (-4)
0.80	-68.479668346					2.828 (-3)
0.90	-6.87629546					5.382 (-5)
1.00	-4.588037825					1.807 (-5)

Table 1.3: Performance of Some Existing Algorithms and Scheme (2.2) on the IVP (4.1). $h = 005$; /Error/

x	Otunta and Nwachukwu (2003)	Otunta and Nwachukwu (2005)	Scheme (2.2)
0.10	4.433006781863448 (-8)	4.460393447050195 (-8)	6.6932607451310176 (-11)
0.20	4.693590886597176 (-8)	4.746470559009062 (-8)	3.0695753110497981 (-10)
0.30	5.216949414990854 (-8)	5. 297316414964577 (-8)	5.5320928064827964 (-10)
0.40	6.160531255811286 (-8)	6.275687152517258 (-8)	8.5137709725439690 (-10)
0.50	7.924770085654036(-8)	8.092770849906523 (-8)	1.2887036997745267 (-9)
0.60	1.1796484401049940 (-7)	1.2067514376316450 (-7)	2.1209308687548111 (-9)
0.65	1.5970307536123510 (-7)	1.6347215052165730 (-7)	2.9668483938906295 (-9)
0.70	2.5123941810701700 (-7)	2.5728283231602810 (-7)	4.7749028466160283 (-9)
0.75	6.0352354492577680 (-7)	6.1821289199100420 (-7)	1.1629583772747506 (-8)
0.80	1.46192721433428100 (-6)	1.49767978700874800 (-6)	1.8009123262503751 (-8)
0.90	1.8782321300181480 (-7)	1.9160611775376290 (-7)	2.9097551628548966 (-7)
1.00	1.0264357027705940 (-7)	1.0461534818915210 (-7)	1.5896134668470297 (-7)

Problem (2)

Evans and Sangui (1987), Otunta and Ikhile (1999), Otunta and Nwachukwu (2003). Otunta and Nwachukwu (2005).

$$y^1 = -y\,,\ y(0) = 1,\ 0 \le x \le 1 \qquad (4.2)$$

Table 2.1: Performance of Some Existing Algorithms on the IVP (4.2): $h = 0.1$ *and* $h = 0.01$, $0 \le x \le 1$, /Error/

	$h = 0.1$			$h = 0.01$
x	RKM	Evans and Sangui (1987)	Otunta and Ikhile (1999)	Otunta and Ikhile (1999)
0.1	8.537769 (-8)	1.993220 (-7)	1.84352654 (-8)	2.375403 (-10)
0.2	1.44768 (-7)	3.509680 (-7)	4.5222131 (-8)	9.206934 (-11)
0.3	2.046044 (-7)	4.844737 (-7)	5.391025 (-8)	1.911267 (-10)
0.4	2.399892 (-7)	5.776375 (-7)	6.153260 (-8)	1.519854 (-10)
0.5	2.755226 (-7)	6.574187 (-7)	6.684795 (-8)	1.539043 (-10)
0.6	2.95112 (-7)	7.121759 (-7)	7.008194 (-8)	7.519851 (-11)
0.7	3.166163 (-7)	7.543546 (-7)	7.208025 (-8)	1.584570 (-10)
0.8	3.270209 (-7)	7.796861 (-7)	7.369261 (-8)	9.454559 (-11)
0.9	3.314901 (-7)	7.922771 (-7)	7.384228 (-8)	1.231553 (-10)
1.0	3.315419 (-7)	7.948056 (-7)	7.406362 (-8)	4.138562 (-11)

Table 2.2: Performance of Some Existing Algorithms and Scheme {2.2} on the iVP (4.2): h = 0.1, h = 0.01 / /'Error/

h = 0-1			
X	**Otunta and Nwachukwu (2003)**	**Otunta and Nwachukwu (2005)**	**Scheme (2.2)**
0.1	1.3659031247654 (-10)	1.28526804578 (-12)	7.1 165199564835491 (-15) i
0.2	2.7318054373566 (-10)	2.57048946047 (-12)	1.4238309376726173 (-14) |
0.3	4.0977087343182 (-10)	3.85571214472 (-12)	2.1 1308850271 1 1965 (-14,
0.4	5.4636113420520 (-10)	5.14085963419 (-12)	2.8322013989241994 (-14)
0.5	6.8295125398181 (-10)	6.42597190571 (-12)	3.5327652497267432 (-14)
0.6	8.1954166037576 (-10)	7.71 131460603 (-12)	4.2279827337136359 (-14) i
0.7	9.5613187240543 (-10) | 8.99640394589 (-12)		4.9073935947085007 (-14)
0.8	1.09272224091033 (-9) 1.028! 56420546 (-1 1)		5.5964679590966526 (-14)
0.9	1.22931260250409 (-9) 1.1566733 12446 (-11)		6.3079354925809201 (-14)
1.0	1.36590294983700 (-9) 1.2851 87429027 (-1 1)		6.9864363548645434 (-14) :
A-0-01			
0.1	9.79024556160 (-12)	0.000000000000 j 1.2269861993937067 (-16) 1	
0.2	1.958052296696 (-11)	0.000000000000	2.7120589289001843 (-16)
0.3	2.937073 107441 (-1 1) j i.498644300000(-16)		1.0490510297147552 (-15)
0.4	3.916089807901 (-11) | 3.312516200000 (-16)		1.6562581280257984 (-15)
0.5	4.895094658387 (-11) ! 1.830448300000 (-16)		2.0134931475125769 (-15)
0.6	5.874144697524 (-11)	4.045916400000 (-16)	2.6298457195346804 (-15)
0.7	6.853158279455 (-11)	5.589286500000 (-16)	2.9064290082554546 (-15)
0.8	7.832176465571 (-11)	6.1771 16800000 (-16)	3.335643 i 544282728 (-15)
0.9	8.81 1188996686 (-11)	5.461415900000 (-16)	3.2768496065353277 (-15)
1.0	9.790215269197 (-11)	6.035798000000 (-16)	3.3 196889807129333 (-15)

Problem (3)

Tam (1989), Ikhlie (1998), Otunta and Nwachukwu (2003), Otunta and Nwachukwu (2005)

$$y^1 = \frac{-y^3}{2}\ ;\quad y(x) = \frac{1}{\sqrt{1+x}},\ y(0) = 1$$

(4.3)

Table (3.1) performance of Ikhile (1998) on the IVP (4.3); /error/

x	h = 0-1	h = 0-01
1.0	5.6968 (- 4)	4.7817 (-6)
2.0	3.8232 (- 4)	3.26078 (-6)
5.0	1.5583 (-4)	1.34904 (-6)
10.0	6.5955 (-5)	5.74678 (-7)
20.0	2.56312 (-5)	2.2408188 (-7)
100.0	2.47837 (- 6)	2.165628 (-8)
1000.0	7.9804 (- 8)	6.915024 (-10)

Table 3.2: Performance of Some Existing Algorithms and Scheme (2.2) on the IVP (4.3);

h = 0 • 1, h = 0 • 01, /Error/

h = 0 ■ 1 ;			
x	**Otunta and Nwachukwu (2003)**	**Otunta and Nwachukwu** (2005)	**Scheme (2.2)**
1.0	5.921 5440479188968(-10)	1.6482675621744727 (-11)	2.57181 14473246850 (-13) 1
2.0	4.1558415788079232 (-10)	1.1291252301779519(-11)	1.7441271565501967 (-13)
5.0	2.1017826834332891 (-10)	5.6680760045 1 34681 (-12)	8.81 1 1 149117089341 (-14) !
10.0	1.1472314798713418 (-10)	3.0908329773545908 (-12)	4.9525498624672143 (-14) i

20.0	6.0097087662289276 (-1 1)	1.6195360698017589 (-12)	2.5819981321785752 (-14)
100.0	1.2499165637232028 (-11)	3.2524413646270637 (-13)	1.1715483474646886 (-14) ■
1000.0	1. 1833032329755037(02)	9.2424983503163548 (-14)	1.1767171296364468 (-13)

h = 0-01			
	Otunta and Nwachukwu (2003)	**Otunta and Nwachukwu**	**Scheme (2.2)**
1.0	5.8093420971300028(05)	4.7102773760513238 (-16)	7.8504622934188707 (0 6)
2.0	1.9229626863835681 (-15)	1.9229626863835648 (-16)	9.6148134319178304 (-16)
5.0	3.8072718754294461 (-15)	8.1584397330631344 (-15)	4.0792198665315506 (-1 0
10.0	6.8120574316460184 (-15)	1.0678360298255961 (-14)	7.3643864125902946 sE O
20.0	6.4867933370001995 (-15)	1.1320090333196372 (-14)	9.6665939923924235 I - EM
100.0	5.7880067166410671 (-14)	5.0488154974074262 (-14)	9.9581609534495!78 i- 4)

Conclusion

The results obtained through our proposed scheme exhibit remarkable improvement over. existing schemes. In addition, the method is L-stable and it resolves singularities. The accuracy in the neighbourhood of the singular point is quite impressive.

References

Evans, J. D. and Sangui, B. B. (1987). A New Fourth Order RKM for I VPs. *Computer; --m, Mathematics II (e*d. S.O. Fatunla) Boole Press (1987).

Fatunla, S. O. (1982). Nonlinear Multistep Methods for I VPs. *Journal Computer Maths Applications* 8, 231-239.

Fatunla, S. O. (1986). Numerical Treatment of Singular/Discontinuous IVPs. *Journal ('ompun ■ Maths with Application* 12,1 10-1 15.

Fatunla, S. O. (1988). *Numerical Methods for IVPs in ODES.* Academic Press, Boston, fUS.A,

Fatunla, S. O. (1990). On the Numerical Solution ofSingular *IVPs ABACUS* 19(2), pp. 121.130

Fatunla, S. O. (1994). Recent Development in the Numerical Treatment of Singular; Discontinuous IVPS. *Scientific Computing* (ed. S. O. Fatunla), Ada + Jane Press, pp. 46-60.

Fatunla, S. O. and Aashikpelokhai, U.S.LJ. (1994). A Fifth order L-stable Numerical Integrate;' *Scientific Computing* (Fatunla, S. O. Ed) pp. 68-86.

Ikhile, M. N. O. (1998). Nonlinear One-Step methods for IVPs in ODEs. Ph.D. Thesis, University of Benin, Benin City, Nigeria.

Lambert, J. D. and Shaw, B. (1965). On the numerical solution of $y' = f(x,y)$ by a class of formulae based on rational approximation, *Math Comp.* 19, 456-462.

Lambert, J. D. (1974). Non linear Methods for stiff system of ODES. *Conference on the Numerical solutions of ODEs* Dundee, 77-88.

Luke, Y. L.; Fair, W. and Wimp, j. (1975). Predictor-Corrector formulas based on Rational interpolants. *Journal Computer and Maths with Applications* 1, 30 2.

Otunta, F. O. and Ikhile, M. N. O. (1996). Stability and Convergence of a Class of Variable order non-linear one-step Rational Integrators of IVPs in ODEs, *Intern. Journal of Computer Mathematics* 62, 199-208.

Otunta, F. O. and Ikhile, M. N. O. (1999). Efficient Rational one-step Numerical Integrators for Initial Value Problems in Ordinary Differential Equations. *Intern. Journal of Computer Mathematics* Vol. 72, pp 49-61.

Otunta, F. O. and Nwachukwu. G. G. (2003). An R [2, 3; 1:5] Rational one-step Integrators for Initial Value Problems in Ordinary Differential Equations, *Nigerian Journal of Mathematics and Applications*, 16(2), pp. 146-158.

Otunta, F. O. and Nwachukwu, G. C. (2005). R [2, 4; 2:6] Rational one-step Numerical Integrator for Initial Value Problems in Ordinary Differential Equations. *Journal of the Nigerian Association of Mathematical Physics*, Vol. 9, pp 285-294.

Tam, H. W. (1989). Parallel methods for the numerical solution of ODEs, Ph.D. Thesis, Rep .No. UIUCDCS-R-98-1516, Dept. ofComp. Science, Univ. of Illinois at Urbana-Champaign.

Van Nieket'k, F. D. (1987). Non linear one-step Methods for IVPs. *Journal Computer and Maths with Applications* 13, 367-371.